Stable vortex dipoles in nonrotating Bose-Einstein condensates

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We find stable families of vortex dipoles in nonrotating Bose-Einstein condensates. The vortex dipoles correspond to topological excited collective states of the condensed atoms. They exist and are dynamically and structurally stable for a broad range of parameters. We show that they can be generated by phase-imprinting techniques on the ground state of condensates.

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Vortices are ubiquitous entities that have fascinated scientists for centuries. They have been observed in almost all branches of physics [1,2], appearing, e.g., as flows in hydrodynamics [1], persistent currents in superfluids [3], nested phase singularities in optical fields [4–8], or vortex lines in Bose-Einstein condensates (BECs) [9–20]. In superfluids, for example, the nucleation of vortex lines assures the dissipation-free rotational motion of the fluid and an intriguing result is the quantization of the circulation of the velocity around these vortex lines. The recent growing interest in the study of BECs, due to the experimental accessibility to this state of condensed matter, has opened new opportunities related to the existence of vortex ensembles hosted in the condensates. Vortices were nested in condensates either by stirring the condensate with a laser [9–16] or by using topological phases in condensates trapped in Joffe-Pritchard magnetic traps [17]. When stirring the condensate the vortices or, to be more precise, the vortex lines that are generated have all the same topological charge. However, recent theoretical works [21–23] have shown that noninteracting condensates could host more complex stationary vortex structures, consisting of vortices of different topological charges such as stable vortex quadrupoles. Stationary vortex quadrupoles were found to exist as excited states in nonrotating symmetric traps in the interacting case indeed [23] and it is believed that a rich variety of vortex-cluster structures do exist in nonrotating BECs.

Recent experiments [15] have revealed that self-assembly of vortices into complex structures (e.g., regular vortex lattices) is a robust feature of BECs. These complex vortex structures are excited collective states of BECs and unlike what happens in optics, where higher order self-sustained structures suffer a variety of instabilities (dynamical, structural, and modulational), they are much more stable than commonly believed. Suitable excited states can be viewed as *atomic soliton clusters*, in the spirit of the soliton molecules made of spatiotemporal optical solitons [24–28] which tend to be unstable or, at best, metastable. In this paper we show that stable, topological two-vortex excited collective states of condensed atoms do exist in nonrotating BECs. These states host two vortices of opposite topological charges thus being termed *vortex dipoles* (VDs).

The model. In the zero-temperature limit the dynamics of the condensed atoms in a magnetically confined BEC is ruled by the Gross-Pitaevskii equation, which, in normalized form reads

$$i\frac{\partial \Psi}{\partial t} = -\frac{1}{2}\Delta\Psi + \frac{1}{2}\sum_{\eta=x,y,z}\omega_\eta^2\eta^2\Psi + U|\Psi|^2\Psi, \qquad (1)$$

where Ψ is the wave function, ω_η are the normalized trap frequencies, and U is an adimensional interaction strength proportional to the scattering length of the atoms in the condensate. The time variable t is normalized with respect to the trap characteristic period and the spatial variables with respect to the trap characteristic length, respectively. Finally $N=\int|\Psi|^2 d^3\vec{r}$ is the number of atoms in the condensate.

Construction of a vortex dipole state. To motivate the subsequent analysis we first construct numerically vortex dipoles on the basis of the full three-dimensional Eq. (1). First, we compute the *ground state* (GS) of a BEC in a pancake-type trap with parameters $\omega_x=\omega_y=1$ and $\omega_z=2$ [Fig. 1(a)], then we imprint the appropriate phase on it, given by $\arg[x^2-2+y+i(x^2-2-y)]$ (this choice will be justified later) and let the condensate evolve in the three-dimensional space up to $t=40$ to study its dynamical stability. The outcome is shown in Figs. 1(b) and 1(c).

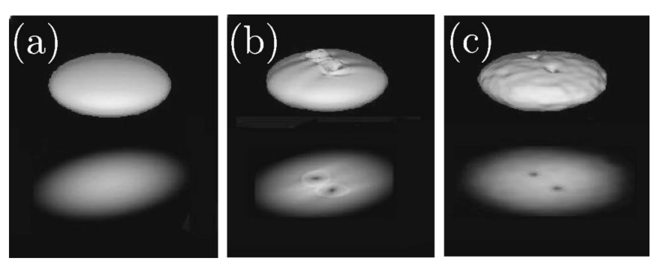

FIG. 1. (a) Ground-state stationary solution of Eq. (1) for $\omega_x=\omega_y=1$, $\omega_z=2$, and $UN\simeq 4160$. (b) Solution shortly after the phase imprinting procedure at $t=0.2$, and (c) solution after $t=40$. Shown are the isosurfaces of the condensate and the integrated views along the z axis.

These results seem to imply that VDs can exist as stable structures in BECs. In what follows we confirm the existence and stability of such topological structures.

Stationary vortex dipole solutions. To simplify the analysis here we concentrate on pancake traps, were the z direction is tightly confined so that the z coordinate can be decoupled and the system becomes two dimensional. Thus $\Psi(x,y,z;t)=\psi(x,y,t)e^{-\omega_z z^2/2}e^{i\omega_z t/2}$ with

$$i\frac{\partial\psi}{\partial t}=-\frac{1}{2}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+\frac{1}{2}\sum_{\eta=x,y}\omega_\eta^2\eta^2\psi+g|\psi|^2\psi, \tag{2}$$

where g is a reduced nonlinear coefficient because of the normalization factors (see, e.g., Refs. [29,30]). From now on we take $N=\int|\psi|^2d^2\vec{r}$. The possibility of finding stable nontrivial structures such as vortex quadrupoles in symmetric traps ($\omega_x=\omega_y$) has been investigated for strongly interacting condensates in Ref. [23]. However, asymmetric traps, i.e., those with $\omega_x\neq\omega_y$, offer more degrees of freedom than the symmetric ones, and stationary vortex structures can be found in such potentials. For example, one can build in the particular case when the ratio $\omega_y/\omega_x=2$, in the noninteracting case ($g=0$), a stationary VD as a linear combination of Hermite polynomials in variables x and y. For $\omega_x=1$ this stationary solution hosting a VD is $\psi_{dip}(t,x,y)=[(4x^2-2-2\sqrt{2}y)+i(4x^2-2+2\sqrt{2}y)]\times\exp(-x^2/2-y^2)\exp(-7it/2)$. Starting from this solution we are going to look for stationary VDs in interacting condensates. Any one-parameter family of stationary solution to Eq. (1) is of the type $\psi(t,x,y)=\varphi_\mu(x,y)\exp(-i\mu t)$, where $\varphi_\mu(x,y)$ is the envelope of the stationary solution corresponding to the chemical potential μ.

We have numerically calculated several types of nonlinear stationary states with the symmetry $\varphi_\mu(-x,y)=\varphi_\mu(x,y)$ by using a Newton relaxation technique: GSs, VDs [see insets in Figs. 2(a) and 2(b)], and *soliton dipoles* (SDs) [see insets in Figs. 2(c) and 2(d)], and VDs [see insets in Figs. 2(c) and 2(d)]. The GS solution, displaying a constant phase, is the one of lowest energy, and is known to be dynamically stable. Soliton dipoles, which are the multidimensional extension of black solitons, are two-humped solutions and display a step-like phase. They are known to be dynamically unstable in the nonlinear Schrödinger equation and their equivalents in three-dimensional geometries were found to disintegrate into vortex rings [13,14]. On the contrary the phase front of the VD solutions is nontrivial and, in the weakly interacting limit their phase gradient displays a decay as $\nabla\Phi\approx1/\rho^2$, Φ being the phase of the complex field and ρ the radial coordinate. This resembles the features of electric charge dipoles.

We have first calculated the density profiles of the stationary VDs in an asymmetric trap with $\omega_x=1$ and $\omega_x=2$ using as initial trial function $\psi_{dip}(t=0,x,y)$. Then we proceed by a continuation method varying slightly both the interaction strength g and the chemical potential μ until the regime of strong interactions $gN=100$ [30] is reached. Finally, we proceed by varying ω_x from $\omega_x=1$ to $\omega_x=2$ while keeping ω_y constant. For all these parameter values we obtain VD solutions proving that they exist for a continuous range of trap frequencies, including the case of symmetric traps. Further, for the symmetric trap, we have decreased the strength of the interaction and we have observed that no VDs exist below a threshold $(gN)_{th}\approx20$. While approaching this threshold, the vortices separate from each other, and at cutoff the VDs degenerate into the unstable SDs. Figure 2 shows the diagrams $E(N)$ and $\mu(N)$ for the two extreme situations corresponding to the symmetric trap with $\omega_x=\omega_y=2$ and the asymmetric trap with $\omega_x=1$ and $\omega_y=2$ for the three types of stationary states mentioned above and additionally for the *vortex solitons* (VSs). One can see, for the case of symmetric trapping potentials, that the branch corresponding to VDs bifurcates from the branch corresponding to unstable SDs. This is in agreement with the absence of stationary VDs in noninteracting symmetric condensates, and is a point that must be emphasized on physical grounds: The vortex dipoles are *purely nonlinear entities* self-sustained by the interactions between the condensed atoms, thus with no counterparts in noninteracting systems. It is remarkable that BECs are the ideal laboratory for finding these nonlinear structures since other systems ruled by nonlinear Schrödinger equations similar to Eq. (1), such as optical systems, have typically smaller interaction strengths.

It is remarkable that the vortex dipole is energetically more favorable than the soliton dipole so that it plays the role of the "second excited state" of this quantum system, this being a purely nonlinear effect coming from the last term in Eq. (2). It is possible to justify this behavior of the nonlinear system by analyzing the energy functional

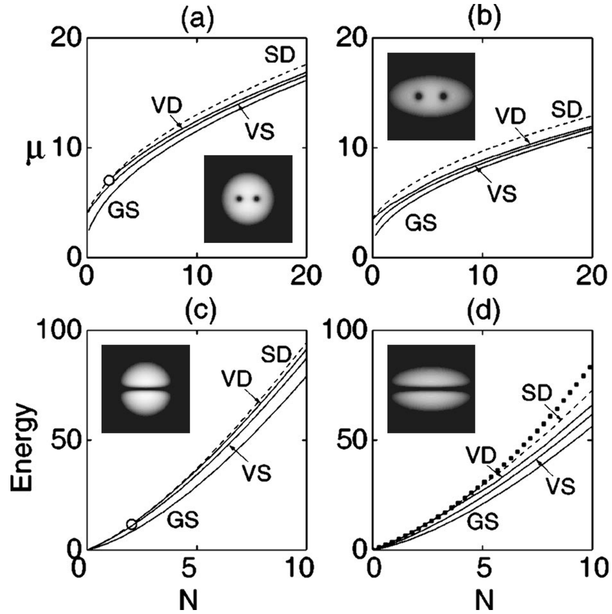

FIG. 2. Chemical potential and energy vs number of atoms for (a), (c) a symmetric trap with $\omega_x=\omega_y=2$ and (b), (d) an asymmetric trap with $\omega_x=1$ and $\omega_y=2$. Here $g=10$. Dashed lines: the soliton dipole branch. Filled squares: the variational approach results for the vortex dipole. The insets show typical density distributions for the vortex dipoles in symmetric traps [panel (a)] and asymmetric traps [panel (b)] and soliton dipoles in symmetric traps [panel (c)] and asymmetric traps [panel (d)].

$$E[\psi]=\int d^2\vec{r}\frac{1}{2}\left[|\nabla\psi|^2+\sum_{\eta=x,y}\omega_\eta^2\eta^2|\psi|^2+g|\psi|^4\right]. \quad (3)$$

Let us look for its extremum under the restriction $\int d^2\vec{r}|\psi|^2 = N$ over the family of trial functions defined as $\psi=\alpha_1\phi_1 + \alpha_2\phi_2$, where $\phi_1=(x^2-1/2\omega_x)\exp(-\omega_x x^2/2-\omega_y y^2/2)$ and $\phi_2=y\exp(-\omega_x x^2/2-\omega_y y^2/2)$. For instance, for $\alpha_1=0$ we have a soliton dipole and for $\alpha_1=1$ and $\alpha_2=i$ we have a vortex dipole, i.e., the complex function ψ hosts two vortices of opposite charges located at the crossing of the lines $x=\pm\sqrt{1/2\omega_x}$ $[\mathrm{Re}(\psi)=\phi_1=0]$ and $y=0$ $[\mathrm{Im}(\psi)=\phi_2=0]$. After minimizing the energy with respect to $\alpha_{1,2}$ taking into account the restriction we obtain that (i) α_1/α_2 should be imaginary, (ii) the energy of "mixed" states with $\alpha_1\neq 0$ and $\alpha_2\neq 0$ is smaller than the energy of any "pure" one, $E[\psi] < E[\phi_{1,2}]$, if the quantity $\delta=[\delta+2N(C_{11}-C_{12})][\delta + 2N(C_{12}-C_{22})]$ is negative. Here $C_{jk}=\langle\phi_j^2\phi_k^2\rangle/\nu_j\nu_k$, $\nu_j = \langle\phi_j^2\rangle$, $\langle\cdots\rangle=\int d^2\vec{r}\cdots$ and $\delta=2\omega_x-\omega_y$. This being the case the optimal ratio between the moduli of $\alpha_{1,2}$ is given by $|\alpha_1|^2:|\alpha_2|^2=\nu_2(C_{22}-C_{12}):\nu_1(C_{11}-C_{12})$. Calculating all integrals in the case $\omega_x=\omega_y/2=1$, one gets $\delta=0$, $C_{11}:C_{12}:C_{22}=41:12:48$ which shows that indeed $\delta<0$, i.e., formation of mixed state (vortex dipole) is profitable. The resulting value of energy can be written as $E=(\gamma g N^2 + 7N/2)$ with $\gamma=57\sqrt{2}/520\pi$. Note that the N dependence of E in Fig. 2 is in agreement with this theoretical prediction.

Stability of vortex dipoles. The study of the dynamics and stability of the vortex dipoles in the presence of perturbations is of paramount importance both from the theoretical and experimental points of view. The fact that these states are excited states does not automatically imply their instability. Theoretical studies have revealed the existence of robust metastable solitons both without or with vorticity (see, e.g., Refs. [31–35]). In the naive analogy between vortices and electrical charges, one might expect that a vortex dipole is not stable, the constituent vortices always annihilating themselves. However, such analogy is only justified in static conditions, and does not hold in dynamical regimes. In particular, it was recently shown that even in the noninteracting case, for a symmetric trap, vortex-antivortex pairs can exhibit rich dynamical features [36]. Namely, depending on the initial distance between the constituent vortices a variety of scenarios were theoretically shown to occur: (i) the vortices move along nonintersecting trajectories, (ii) they periodically annihilate themselves and revive after a while flipping the vorticity, and, for a critical separation between vortices, and (iii) they periodically flip the topological charge [36]. All such features can be understood in terms of the so-called Berry vortex trajectories [37] in the framework put forward by Freund [38]. The nonlinear interaction brings even more complexity into play. In some sense, in the strongly nonlinear regime the vortices become like a small structure and the nonlinearity generates a kind of barrier, both physical and energetic, that might prevent the recombination of the topological objects.

We investigated the stability of the VDs against two different kinds of perturbations. First, the dynamical instabilities, i.e., instabilities of the solutions under small perturbations to the initial data. Second, the structural instabilities which may develop as a consequence of perturbations performed on the model, e.g., when changing the trap frequencies [30] or the strength of the interaction.

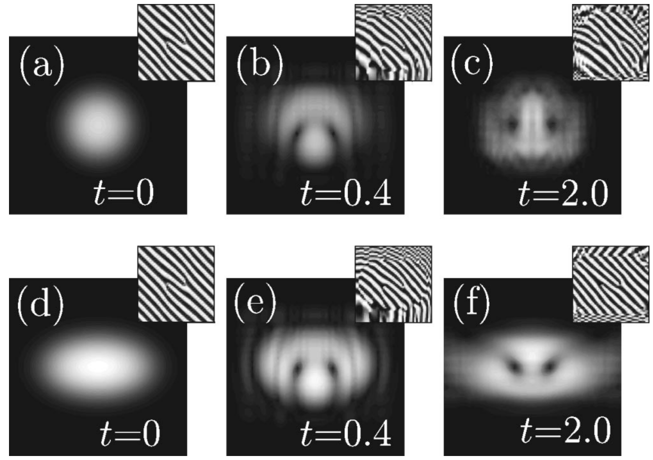

FIG. 3. (a)–(c) Charge flipping in weakly interacting condensates ($gN\approx 16.5$) for $\omega_x=\omega_y=2$. (d)–(f) Generation of a vortex dipole by phase imprinting (see text) in a trap with $\omega_x=1$, $\omega_y=2$, and $gN\approx 20.5$. Shown are density plots and interference fringes (see Ref. [30]) for $(x,y)\in[-3,3]\times[-3,3]$.

To elucidate the dynamical stability of the VDs, we have conducted numerical experiments by solving Eq. (2) taking as initial data a VD with different types of random perturbations imposed. In all the situations analyzed we have observed that the VDs are extremely robust with respect to initial noisy perturbations up to the maximum times used (of the order of $t=1000$). This is remarkable since a naive intuition might suggest that a vortex-antivortex pair should have a strong tendency to recombine into a simpler zero-topological charge configuration. However, not only is this not the case but vortex dipoles also behave like strong attractors of the system. Our simulations reveal that once an initial distribution with a phase distribution close to that of the VD but with a completely different density profile, is launched into the trap, it will evolve towards the VD state. Similar to what we have done in the three-dimensional situation shown in Fig. 1, we have imprinted on the ground states of the condensate phase masks corresponding to two vortices with opposite vorticities. In a symmetric trap, if the ground state correspond to a chemical potential for which no VD solutions exist, i.e., in the weakly interacting limit, no stable VD is generated [see, e.g., Figs. 3(a)–3(c)]. The dynamics resembles that of a VD in a symmetric trap in the noninteracting case, when then vortices flip the charges [36]. If the trap is asymmetric, even in the weakly interacting regime, one can generate VDs by imprinting on the ground state of the condensate an adequate phase mask. In this case, even though pulsating dynamics of the cloud is visible, no charge flipping occurs [see Figs. 3(d)–3(f)] and the VD survives. In the strongly interacting regime, in both symmetric and asymmetric traps, the VD generated does not display any charge flipping showing robustness on propagation.

To study the structural stability of VDs we have performed, as indicated above, a sharp change of the trap fre-

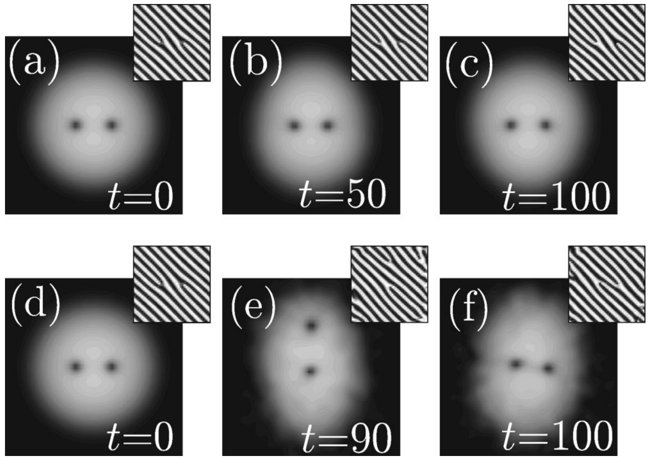

FIG. 4. Effect of structural perturbation on the vortex dipole. (a)–(c) Robust propagation of the symmetric vortex dipole corresponding to $\omega_{x,y}=2$ and $gN\approx 190$ in a trap with $\omega_x^2=4.4$, $\omega_y^2=3.6$. (d)–(f) Charge flipping induced by a strong structural perturbation. The initial state is the same as in panel (a) and the trap frequencies are $\omega_x^2=5$ and $\omega_y^2=3$.

quencies by increasing ω_x with $\Delta\omega_x$ and decreasing ω_y with $\Delta\omega_y$ and followed the evolution. We have observed that the VD survives such strong perturbations, provided $\Delta\omega_{x,y}$ are moderate, i.e., $\Delta\omega_x/\omega_x \lesssim 5\%$. In this case, the atom cloud hosting the VD pulsates transversally but the vortices perform only small oscillations around their equilibrium positions, as shown in Figs. 4(a)–4(c) without topological charge flipping. As expected with large changes of the trap frequencies, e.g., $\Delta\omega_x/\omega_x > 10\%$, the VD cannot survive, and, via extremely sharp Berry trajectories [37], the vortices exchange the charges [see Figs. 4(d)–4(f)].

In conclusion, we have found one-parameter families of vortex dipoles in tightly confined nonrotating symmetric and asymmetric BECs. Such dipoles correspond to purely nonlinear collective excited states which exist in a wide range of the parameters, do not have counterparts in the noninteracting limit in symmetric traps, and are extremely robust under initial perturbations and even survive moderate structural perturbations. Our numerical experiments reveal that the vortex dipoles could be generated by imprinting an adequate phase mask on a condensate residing in the ground state both in two-dimensional system and in the more realistic three-dimensional system [39–42]. In particular, the possibility of phase imprinting of vortex-antivortex pairs in toroidal trapped condensate has been already shown numerically [43]. The topological phase technique proposed in Ref. [17] might be more suitable for generation of the vortex dipole than the phase-imprinting technique that might face serious difficulties. However, it is noteworthy to mention the rapid advance in the phase-imprinting technique, mainly motivated by the generation of the holographic optical traps. Another difficult task from the experimental point of view is the choice of an adequate detection technique in order to avoid the destruction of the vortex-dipole state that can occur if the atomic cloud is let to freely expand. A solution to this problem would be the use of an interferometric method that was successfully used to detect a single vortex nested in a BEC [44].

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