Institute of Physics Publishing — Journal of Optics B: Quantum and Semiclassical Optics

J. Opt. B: Quantum Semiclass. Opt. **6** (2004) S333–S340 — PII: S1464-4266(04)71102-7

Soliton clusters in three-dimensional media with competing cubic and quintic nonlinearities

D Mihalache[1,2,3], D Mazilu[1,2], L-C Crasovan[3], B A Malomed[4], F Lederer[2] and L Torner[3]

[1] Department of Theoretical Physics, Institute of Atomic Physics, PO Box MG-6, Bucharest, Romania
[2] Institute of Solid State Theory and Theoretical Optics, Friedrich-Schiller-Universität Jena, Max-Wien-Platz 1, D-07743, Jena, Germany
[3] ICFO—Institut de Ciencies Fotoniques and Department of Signal Theory and Communications, Universitat Politecnica de Catalunya, ES 8034 Barcelona, Spain
[4] Department of Interdisciplinary Studies, Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel

Received 29 October 2003, accepted for publication 9 December 2003
Published 4 May 2004
Online at stacks.iop.org/JOptB/6/S333
DOI: 10.1088/1464-4266/6/5/023

Abstract
We introduce a class of robust soliton clusters composed of N fundamental solitons in three-dimensional media combining the self-focusing cubic and self-defocusing quintic nonlinearities. The angular momentum is lent to the initial cluster through staircase or continuous ramp-like phase distribution. Formation of these clusters is predicted analytically, by calculating an effective interaction Hamiltonian H_{int}. If a minimum of H_{int} is found, direct three-dimensional simulations demonstrate that, when the initial pattern is close to the predicted equilibrium size, a very robust rotating cluster does indeed exist, featuring persistent oscillations around the equilibrium configuration (clusters composed of $N = 4$, 5, and 6 fundamental solitons are investigated in detail). If a strong random noise is added to the initial configuration, the cluster eventually develops instability, either splitting into several fundamental solitons or fusing into a nearly axisymmetric vortex torus. These outcomes match the stability or instability of the three-dimensional vortex solitons with the same energy and spin; in particular, the number of the fragments in the case of the break-up is different from the number of solitons in the original cluster, being instead determined by the dominant mode of the azimuthal instability of the corresponding vortex soliton. The initial form of the phase distribution is important too: under the action of the noise, the cluster with the built-in staircase-like phase profile features azimuthal instability, while the one with the continuous distribution fuses into a vortex torus.

Keywords: spatiotemporal solitons, light bullets, soliton clusters, spinning light bullets

1. Introduction

Modern nonlinear optics classifies solitons, i.e., self-supporting localized light pulses and beams, as spatial, temporal, or spatiotemporal, depending on whether the self-confinement of the light propagation is observed in space, time, or in both space and time. In the course of the last three decades, they have been predicted and observed in various settings [1–3], including one-dimensional (1D) temporal solitons in optical fibres, 1D and two-dimensional

(2D) spatial solitons (self-localized light beams) in planar and bulk waveguides, as well as 2D and three-dimensional (3D) spatiotemporal solitons (alias *light bullets* [4]) in planar and bulk dispersive optical media. As regards the latter type, the only kinds of spatiotemporal solitons that have thus far been created in a real experiment are quasi-2D ones in bulk quadratically nonlinear media [5] (for detailed analysis of the existence and stability of light bullets in quadratically nonlinear media, see [6]). Several other kinds of optical solitons were discovered, chiefly during the last decade, such as Bragg solitons, vortex solitons, vectorial solitons, discrete solitons, cavity solitons, and photorefractive and holographic solitons [1].

A great deal of recent studies were focused on nonlinear self-trapped optical beams carrying phase dislocations, which are associated with zero-intensity points (for recent reviews of linear and nonlinear *singular optics*, which studies optical fields carrying topological defects, see, e.g., [7]). Typical examples of nonlinear 'singular beams' are vortex solitons [8], multiple vector solitons [9], multi-soliton necklace patterns [10, 11], and soliton clusters [12]. Complex structures in self-focusing nonlinear media, composed of several interacting solitons in the form of ring-like necklaces, have been recently investigated as nontrivial examples of self-trapped states of light carrying phase dislocations [10, 11]. Two-dimensional soliton clusters in saturable self-focusing media, which were recently introduced in [12], are ring-like soliton complexes in bulk media with a staircase-like phase distribution that induces a nonzero angular momentum leading to rotation of the cluster. They are generally metastable (in the absence of random perturbations they can propagate over many diffraction lengths), eventually featuring a symmetry-breaking instability. However, random perturbations destabilize the cluster after passing only a few diffraction lengths, and it eventually disintegrates into a set of isolated 2D solitons.

Soliton clusters in two and three dimensions, respectively, may be viewed as a generalization of the 2D bright vortex solitons (*vortex rings*) [13–18] and spinning light bullets (*vortex tori*) [19–23]. The concept of soliton clusters has also been introduced in the study of nonlinear pattern formation in dissipative media, such as externally driven optical cavities, the simplest example being a 2D clustered pattern observed in the transverse plane [24].

These objects are related to ones with more sophisticated topological properties, such as *skyrmions* (named by analogy with localized structures in the Skyrme model of nuclear forces). The skyrmions are complex localized structures characterized by two topological invariants, which are believed to play an important role in various physical settings, from quantum Hall systems, nematic liquid crystals, and magnetic semiconductors, to Bose–Einstein condensates (BECs). Here we only mention the prediction of skyrmions [25] in a two-component BEC [26] and the proof that they can be energetically stable in a trapped two-component atomic BEC, under realistic experimental conditions [27].

In the rapidly developing field of matter waves, considerable progress has been reported in the generation of vortices [28] in two-component [29] and stirred [30] BECs, the observation of a regular triangular vortex lattice in rotating BECs [31, 32], formation of bright-soliton trains in a quasi-1D optical trap [33], the prediction of bright solitons that can be stabilized in the free 2D space by making the nonlinearity strength oscillate (by means of the Feshbach resonance) between positive and negative values [34, 35], the existence of stable 2D and 3D solitons (including those carrying topological charge) in self-attractive BECs trapped in optical lattices [36], and the prediction of dynamically and structurally stable families of vortex dipoles in nonrotating BECs (corresponding to some topological excited collective states of the condensed atoms) [37]. Still earlier works on vortices and solitons in BECs were reviewed in [38]. Closely related topics are the recently introduced concepts of globally linked vortex clusters in nonrotating BECs with attractive interactions [39], of ring dark solitons and vortex necklaces in BECs [40], and of the so-called soliton molecules in optics [41–43] and mixed atomic–molecular BECs [44–51].

In media with a simple quadratic or cubic nonlinearity, soliton clusters always tend to self-destroy through expansion or collapse, or, at best, they exist as metastable states which are broken up by small perturbations [10–12, 52–55]. However, in the presence of two competing optical nonlinearities (self-focusing and self-defocusing ones), the instability may be greatly suppressed, and the soliton complexes may propagate stably over an extremely large distance even in the presence of random perturbations. The first example of the formation of both 2D [56] and 3D [41] robust soliton clusters was demonstrated in the case of competing quadratic and self-defocusing cubic nonlinearities; these soliton complexes carry nonzero orbital momentum and are linked via a staircase-like phase distribution. A fact which helps to understand the stabilization of these clusters is the existence of stable 2D vortex solitons [57] and 3D vortex tori [21] in the same media; the clusters may be regarded as, roughly speaking, fragmented counterparts of these stable objects.

Recently, it was found that similar stable 2D and 3D bright spinning solitons also exist in media with competing self-focusing cubic and self-defocusing quintic nonlinearities [58–63, 20], which suggests the possibility of investigating quasi-stable soliton clusters in the multidimensional nonlinear Schrödinger (NLS) models with competing self-focusing cubic and self-defocusing quintic nonlinearities. In a recent work [64], we have shown that such complex robust 2D solitonic structures are indeed generic objects in this medium.

A challenge is to construct quasi-stable soliton clusters, composed of a predefined number of initially separated fundamental nonspinning solitons, in the *three-dimensional* cubic–quintic (CQ) model. They may be regarded as 'molecules of lights', if separate fundamental solitons are given the role of 'light atoms'. As regards the possibility of experimental observation of the 3D soliton clusters, it was recently concluded that the dielectric response of several different media may be modelled reasonably well by the CQ nonlinearity (however, complicated by significant two-photon absorption) [65].

Search for quasi-stable soliton clusters in the 3D CQ model is the objective of this work. Necklace-like 3D clusters, composed of several individual solitons, are constructed in section 2. Direct numerical simulations of their propagation, which proves that they are indeed robust complexes, are presented in section 3. The results are summarized in the concluding section.

2. Construction of three-dimensional soliton clusters in media with cubic–quintic nonlinearity

The equation governing the field evolution is the NLS equation of the CQ type, written in a normalized form:

$$\mathrm{i}u_Z + u_{XX} + u_{YY} + u_{TT} + |u|^2u - \alpha|u|^4u = 0, \quad (1)$$

where α is a parameter which characterizes the strength of the quintic nonlinearity that can be scaled out (see below). For earlier works treating the model in the context of nonlinear optics, see, for example, [66]. In the most typical case, equation (1) governs the spatiotemporal evolution of the complex amplitude of the electromagnetic wave along the axis Z in a bulk dispersive medium (X and Y are the corresponding transverse coordinates); soliton solutions to be found in this case then represent 3D spatiotemporal solitons, i.e., self-trapped light pulses in the bulk optical medium.

We now briefly revisit the recently investigated problem of constructing spinning-soliton solutions to equation (1) [22, 23], which are looked for in the form

$$u = U(r, T)\exp(\mathrm{i}S\theta)\exp(\mathrm{i}\kappa Z), \quad (2)$$

where r and θ are the polar coordinates in the (X, Y) plane, κ is the wavenumber shift (the propagation constant), which parametrizes the family of stationary solutions, and the integer S is the above-mentioned spin. The amplitude U may be assumed real, and it obeys the equation

$$U_{rr} + r^{-1}U_r - S^2r^{-2}U + U_{TT} - \kappa U + U^3 - \alpha U^5 = 0. \quad (3)$$

The existence region for the 3D solitons that are sought for in the form of equation (2) is $0 < \kappa < \kappa^{(3D)}_{\mathrm{offset}} = 0.1875$, regardless of the value of the spin and dimension [22, 23].

Equation (1) conserves a dynamical invariant, which has the meaning of the energy of the light pulse:

$$E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u(X, Y, T)|^2\, \mathrm{d}X\, \mathrm{d}Y\, \mathrm{d}T. \quad (4)$$

Other dynamical invariants are the Hamiltonian,

$$H = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [|u_X|^2 + |u_Y|^2 + |u_T|^2 - (1/2)|u|^4 + (1/3)\alpha|u|^6]\, \mathrm{d}X\, \mathrm{d}Y\, \mathrm{d}T, \quad (5)$$

the momentum (equal to zero for the solutions considered in this work), and the z-component of the angular momentum,

$$L_z = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\partial\phi/\partial\theta)\, |u|^2\, \mathrm{d}X\, \mathrm{d}Y\, \mathrm{d}T, \quad (6)$$

ϕ being the phase of the complex field u. Using equation (3), one can readily deduce the relations $L_z = SE$ and

$$H = \kappa E - (4\pi/3)\alpha \int_0^{\infty}\int_{-\infty}^{\infty} U^6(r, T) r\, \mathrm{d}r\, \mathrm{d}T, \quad (7)$$

which are valid for the stationary spinning solitons.

Notice that equation (1) is invariant against rescaling, $\alpha \to \tilde{\alpha} \equiv \lambda\alpha$, $\tilde{Z} = \lambda Z$, $\tilde{X} = \lambda^{1/2}X$, $\tilde{Y} = \lambda^{1/2}Y$, $\tilde{T} = \lambda^{1/2}T$, $\tilde{U} = \lambda^{-1/2}U$, where λ is an arbitrary positive scaling factor. This leads to the corresponding scaling of κ, E, and H:

$$\tilde{\kappa} = \kappa/\lambda; \qquad \tilde{E} = \lambda^{1/2}E; \qquad \tilde{H} = H/\lambda^{1/2}. \quad (8)$$

For the numerical simulations, we set $\alpha = 0.2$.

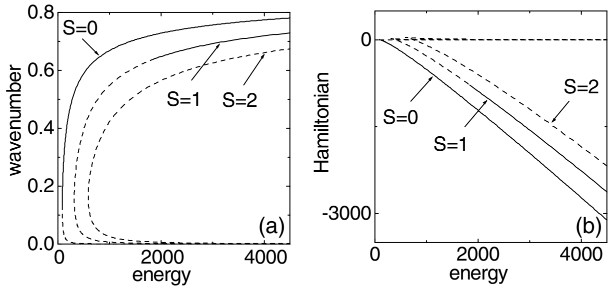

Figure 1. The propagation constant κ (a) and Hamiltonian H (b) of the fundamental and spinning three-dimensional solitons in the cubic–quintic model versus the energy E. Here and below, the scaling factor α in front of the cubic term in equation (1) is fixed to 0.2.

One-parameter families of 3D spinning solutions can be obtained in a numerical form, using a standard band-matrix algorithm to deal with the resulting two-point boundary value problem. The solitons, as expected, have the form of a *vortex torus* with a hole in the centre (since the field must vanish as $r^{|S|}$ for $r \to 0$). In accordance with the results predicted by means of the semi-analytical variational approximation developed in [22], the solutions exist provided that their energy exceeds a certain threshold value. To quantify the 3D solitons, in figure 1 we show the wavenumber κ and the Hamiltonian H for the solitons with spin $S = 0$, 1, and 2 as functions of their energy E. In this figure, continuous and dashed curves correspond to stable and unstable branches (see [20] for further details). Note that κ monotonically increases with E, showing saturation (to the above-mentioned limiting value κ_{offset}) at large values of E. We also see that the threshold energy for the soliton existence drastically increases with S.

In what follows we construct soliton clusters composed of several fundamental (nonspinning) 3D solitons. A simple structure of such a kind is one forming a circular necklace, which is a set of N fundamental solitons set along a circumference of some radius R_0, with a fixed phase difference between adjacent solitons, so that the overall phase change along the circumference is $2\pi M$, where the integer M is the net topological charge of the necklace [12].

Thus, the initial ansatz is

$$u(Z = 0) = \sum_{n=1}^{N} U_0(|\mathbf{r} - \mathbf{r}_n|, T)\mathrm{e}^{\mathrm{i}\phi_n}, \quad (9)$$

where U_0 is the stationary fundamental soliton, $\mathbf{r}_n$ are positions of the soliton centres chosen as specified above, and the soliton phases at these points are $\phi_n = 2n\pi M/N$. The parameters that control the dynamics of the soliton complexes are the topological charge M, the number of solitons N in the set, its initial radius R_0, and the energy E of each constituent soliton.

Note that the ansatz (9) implies that the phase distribution in the initial cluster has a staircase-like form. Below, we will also consider another possibility, with a continuous initial phase distribution that has a form of a ramp with a constant slope,

$$u(Z = 0) = \sum_{n=1}^{N} U_0(|\mathbf{r} - \mathbf{r}_n|, T)\mathrm{e}^{\mathrm{i}M\theta}, \quad (10)$$

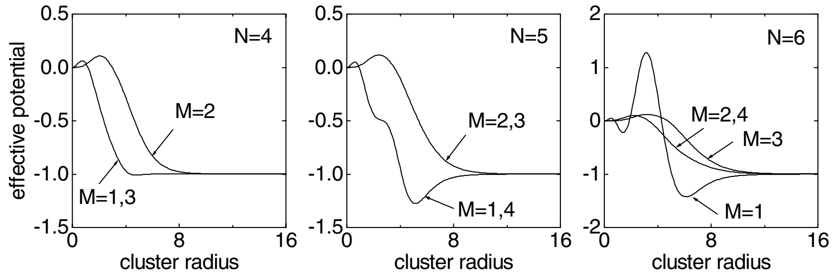

Figure 2. The effective interaction potential versus the radius for the clusters described by the ansatz (9) composed of N fundamental solitons, at different values of the vorticity carried by the soliton complex.

where M is, as above, the net vorticity, and θ is the angular coordinate in the (X, Y) plane.

Recall that the 3D vortex solitons with spin $S = 1$ are stable in the present model if their energy exceeds a threshold value (that is, if they are sufficiently broad) [20]. We therefore focus on consideration of the necklaces whose initial net energy exceeds the corresponding stability threshold value of the $S = 1$ vortex soliton (which is $E_{\text{thr}} \approx 1840$ with the scaling factor $\alpha = 0.2$; see figure 1), anticipating that, for smaller values of the energy, the necklace has no chance to be stable. Thus, we have considered, in particular, the clusters with the net topological charge $M = 1$, composed of $N = 4$, 5, and 6 fundamental solitons, each having the energy of $E = 638$, so that their net energy is well above the threshold.

For the ansatz (9), the cluster's interaction Hamiltonian (alias the effective potential of the interaction), defined as $H_{\text{int}} = H(R_0)/|H(R_0 = \infty)|$, was computed as a function of R_0 and M. This quantity gives important clues concerning the existence and stability of bound states of solitons (see [67] and [12]). The result is that, for the ansatz (9) with $N = 4$, 5, and 6, the interaction Hamiltonian for $M = 1$ displays global minima (as well as for some other values of M due to the symmetry of the physical system under consideration), see figure 2, which strongly suggests the existence of quasi-stable 3D necklace-like patterns.

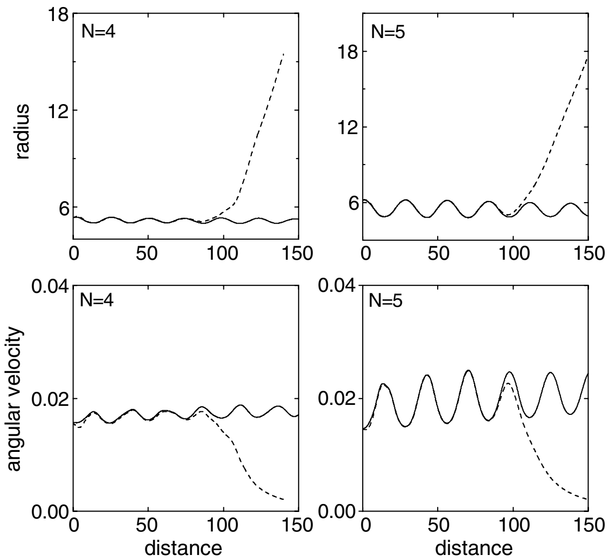

Figure 3. The evolution of the mean radius (top row) and mean angular velocity (bottom row) for clusters composed of four and five solitons. Dashed curves correspond to the case when a random noise is superimposed at input, whereas full curves show the evolution without explicitly added noise. All the soliton complexes in this plot and in the following ones are built up of solitons with the energy $E = 638$. Here the staircase-like phase distribution at input was used.

3. Robustness of soliton clusters in the cubic–quintic medium

By using the predictions following from the computation of the effective potential, we directly simulated equations (1) by means of a finite-difference scheme based on the classical Crank–Nicholson discretization algorithm. This scheme was supplemented by the Newton–Picard iterations and the Gauss–Seidel method for solving the resulting linear system of equations. To achieve good convergence, we needed, typically, five Picard iterations and eight Gauss–Seidel iterations. In most cases, we employed the transverse-grid step size $\Delta X = \Delta Y = \Delta T = 0.3$, and the longitudinal step size $\Delta Z = 0.003$. Transparent boundary conditions allowing the radiation to escape from the computation window were implemented, to prevent possible artificial effects caused by radiation waves re-entering the integration domain.

To present the results in a compact form, we define the cluster's mean radius

$$R(Z) \equiv \frac{1}{E}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(X^2+Y^2)^{1/2}|u|^2\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}T, \quad (11)$$

and its mean angular velocity,

$$\omega(Z) \equiv L_z/I,$$
$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(X^2+Y^2)|u|^2\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}T, \quad (12)$$

where L_z is the z-component of the angular momentum, see equation (6), and I is the cluster's moment of inertia. If the initial radius R_0 of the cluster built as per equation (9) or (10) is large, it is easy to see that the initial value of the average radius (11) amounts to $R(0) \approx R_0$.

Keeping the value of the net topological charge $M = 1$ fixed, we varied the initial radius R_0 around the value corresponding to the potential minimum as predicted by figure 2. In this way, the simulations have produced a range of optimum values of R_0 that minimize oscillations of the mean radius in the course of the propagation (Z-evolution), which implies that the cluster is a nearly stationary state. For $N = 6$, the optimum value is close to $R_0 = 6$, for $N = 5$ it is about $R_0 = 5$, whereas for $N = 4$ we have found $R_0 \approx 4.5$.

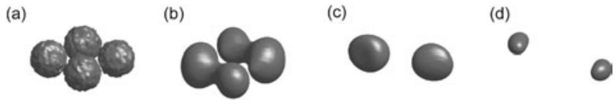

Figure 4. The break-up of the cluster composed of four solitons under the action of random noise: (a) $Z = 0$, (b) $Z = 83$, (c) $Z = 111$, (d) $Z = 140$. Here $M = 1$, $R_0 = 5$, and the input staircase-like phase distribution was used.

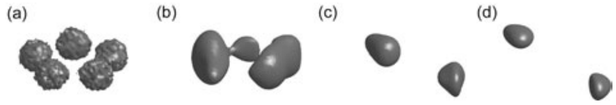

Figure 5. The break-up of a cluster composed of five solitons under the action of random noise: (a) $Z = 0$, (b) $Z = 100$, (c) $Z = 125$, (d) $Z = 150$. Here $M = 1$, $R_0 = 6$, and the initial staircase-like phase distribution was used.

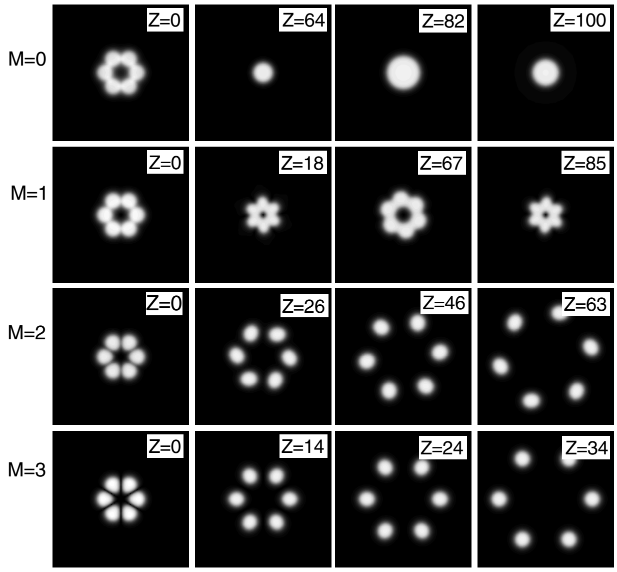

Figure 6. Evolution of the soliton clusters (without the noise added) with $N = 6$, $R_0 = 7$, and different values of the net vorticity M. Here the staircase-like phase distribution at input was used.

In figure 3 we display the evolution of the mean radius $R(Z)$ of the necklace and the corresponding mean angular velocity $\omega(Z)$ in the course of passing several tens diffraction lengths for clusters composed of $N = 4$ and 5 solitons. Due to the fact that in both cases the interaction Hamiltonian exhibits a global minimum (for $M = 1$), although it is less pronounced in the case of $N = 4$, the evolution is quasiperiodic in the absence of an explicitly added noise (see the full curves in figure 3). However, if the noise is injected at input, the cluster eventually breaks up into fragments, after having passed many tens of diffractions lengths in a nearly intact form (dashed curves in figure 3). Typical examples of the noise-induced break-up for the clusters composed of $N = 4$ and 5 solitons are shown in figures 4 and 5, respectively. The remarkable feature of this eventual instability is that the number of the finally emerging solitons is determined mainly by the topological charge M, and *not* by the initial number of solitons, similar to what happens in the case of the azimuthal instability of the corresponding 3D vortex tori [20]. In other words, the final number of fundamental solitons into which the clusters breaks up is determined by the angular 'quantum number' (index) of the azimuthal instability with the largest growth rate of the corresponding vortex torus (soliton). In most cases, the latter number is twice the original spin of the soliton (see figures 4 and 5, where the clusters break up into *two* fragments, despite the fact that they were initially built of *four* and *five* solitons, respectively).

Figures 6 and 7 show the typical evolution of the necklace pattern composed of $N = 6$ solitons without the noise added. The evolution of these clusters is nonstationary: the soliton complex gradually fuses or expands, simultaneously rotating during the evolution.

In the special case of $M = nN$ ($n = 0, 1, 2, \ldots$), the soliton cluster actually carries zero angular momentum, as the phase shifts between adjacent solitons are a multiple of 2π; therefore, in this case the solitons attract each other and the cluster fuses into a single fundamental (nonspinning) soliton (see the first row in figure 6 and the first column in figure 7, corresponding to $N = 6$ and $M = 0$). The soliton system actually has zero angular momentum also for even N and $M = (2n + 1)N/2$ ($n = 0, 1, 2 \ldots$). However, in this case the phase difference between adjacent solitons is tantamount to π; hence the interaction between them is repulsive (see figure 2). Therefore, the cluster gradually expands in this case (see the fourth row in figure 6 and the fourth column in figure 7, corresponding to $N = 6$ and $M = 3$).

When $N = 6$ and $M = 2$, the cluster has a true nonzero angular momentum and the propagation shows gradual expansion and rotation (see figures 6 and 7). When $M = 1$ and $N = 6$, the potential is attractive (see figure 2); however, the nonzero net angular momentum of the structure prevents fusion of the solitons. In such cases, the generic behaviour of the system is its quasiperiodic expansion and shrinkage, which persists over tens of diffraction lengths, as is shown in the second row in figure 6 and the second column in figure 7. This case may be naturally categorized as a truly robust one.

For $N = 6$, $M = 1$, and input radius $R_0 = 7$ (a value slightly larger than that corresponding to the minimum of the interaction Hamiltonian in this case; see figure 2), we compared the long-scale evolution of the clusters perturbed by the random noise in the cases when the initial phase distribution had the staircase and ramped shapes; see equations (9) and (10). Figure 8 shows a comparison of the evolution of the cluster's mean radius and angular velocity for these two different types of input. In both cases, we see pulsating evolution of the soliton clusters over many diffraction lengths; however, in the case of the staircase-like shape of the initial phase mask, the soliton complex eventually breaks up. Examples of the full evolutions are displayed, for the staircase-like and ramp-like profiles and identical initial intensity distributions, in figures 9 and 10, respectively. Thus, the difference between the two types of initial phase distribution is crucially important: when the initial phase profile has the staircase-like form, the soliton cluster is subject to azimuthal instability, the number of emerging fragments being exactly twice the topological charge (recall that $M = 1$ in this case); see figure 9. However, when the initial phase distribution is a ramp-like one, the soliton complex shows a clear trend of slow fusion into a quasi-uniform vortex torus, i.e., a stable 3D spinning soliton, with the same value

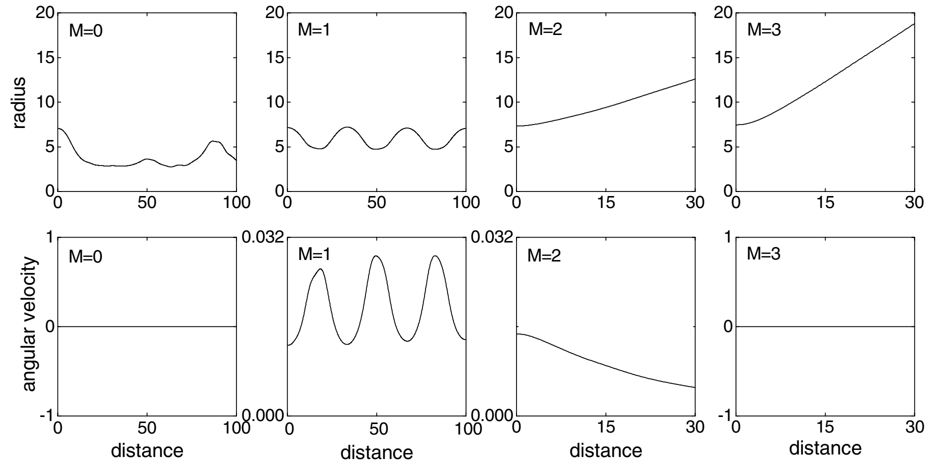

Figure 7. Evolution of the cluster's mean radius and angular velocity (the top and bottom rows, respectively) for different values of the vorticity M. The parameters and conditions are as in figure 6.

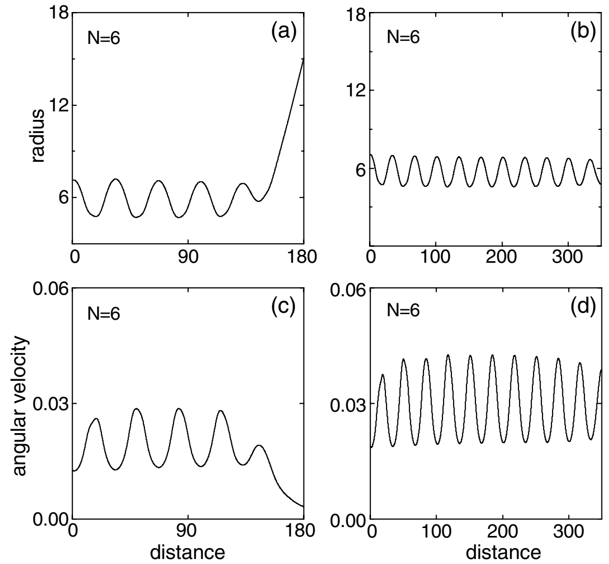

Figure 8. Evolution of the cluster's mean radius and angular velocity (the top and bottom rows, respectively) for the cluster composed of six solitons with the initial radius $R_0 = 7$, random noise being added at input. In (a) and (c), the initial staircase-like phase distribution was used, whereas in (b) and (d), it was the ramp-like distribution.

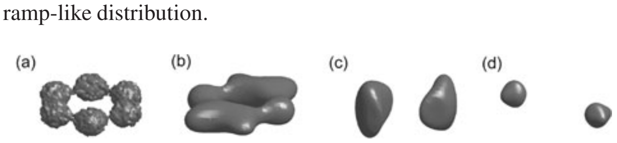

Figure 9. Evolution of the cluster composed of six solitons in the presence of input noise: (a) $Z = 0$, (b) $Z = 125$, (c) $Z = 150$, (d) $Z = 180$. The net vorticity is $M = 1$, the initial radius is $R_0 = 7$, and the staircase-like phase distribution at input was used.

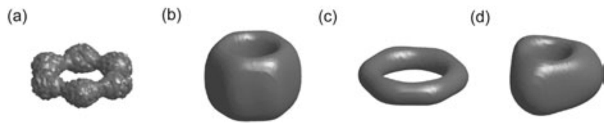

Figure 10. The same as figure 9, but with the continuous ramp-like phase distribution at input: (a) $Z = 0$, (b) $Z = 117$, (c) $Z = 201$, (d) $Z = 350$.

of the vorticity ($S = 1$) which the original soliton cluster was lent (see figure 10).

4. Conclusion

In this work, we have investigated the dynamics of soliton clusters composed of N fundamental solitons in the three-dimensional model combining the self-focusing cubic and self-defocusing quintic nonlinearities. The angular momentum is lent to the initial cluster by means of staircase-like or continuous ramp-like phase distributions. The possibility of formation of robust clusters was predicted in a semi-analytical form, by calculating their effective interaction Hamiltonian. In the cases when minima of the interaction Hamiltonian were predicted, direct three-dimensional simulations demonstrate very long quasi-stable evolution of the rotating clusters, featuring persistent oscillations. If sufficiently strong random noise is added to the initial configuration, the cluster eventually develops an instability, either splitting into several fragments (which are separating fundamental solitons) or fusing into a nearly axisymmetric vortex torus. These outcomes accurately match the stability or instability of the three-dimensional vortex solitons with the same energy and spin; in particular, in the case of the break-up, the eventual number of the fragments is completely different from the number of solitons in the original cluster, being determined by the dominant mode of the azimuthal instability of the corresponding vortex soliton. The initial form of the phase distribution also produces a major effect: if appreciable noise is added, the cluster built with the staircase phase profile features azimuthal instability, while the one with the continuous distribution tends to fuse into a vortex torus with the same value of the vorticity as the original cluster was given. Thus, the recently investigated stable and unstable three-dimensional spinning solitons have their counterparts, with rich intrinsic dynamics, in the form of rotating clusters.

Acknowledgments

Support from Deutsche Forschungsgemeinschaft (DFG), Germany, and from the Institució Catalana de Recerca i Estudis Avançats (ICREA), Spain, is acknowledged. LT acknowledges support by the Generalitat de Catalunya, and by the Spanish Government through grant TIC2000-1010.

References

[1] Kivshar Yu S and Agrawal G P 2003 *Optical Solitons: from Fibers to Photonic Crystals* (San Diego, CA: Academic)
[2] Stegeman G I, Christodoulides D N and Segev M 2000 *IEEE J. Sel. Top. Quantum. Electron.* **6** 1419
[3] Akhmediev N N and Ankiewicz A 1997 *Solitons: Nonlinear Pulses and Beams* (London: Chapman and Hall)
[4] Silberberg Y 1990 *Opt. Lett.* **15** 1282
[5] Liu X, Qian L J and Wise F W 1999 *Phys. Rev. Lett.* **82** 4631
Liu X, Beckwitt K and Wise F 2000 *Phys. Rev.* E **62** 1328
[6] Malomed B A, Drummond P, He H, Berntson A, Anderson D and Lisak M 1997 *Phys. Rev.* E **56** 4725
Skryabin D V and Firth W J 1998 *Opt. Commun.* **148** 79
Mihalache D, Mazilu D, Malomed B A and Torner L 1998 *Opt. Commun.* **152** 365
Mihalache D, Mazilu D, Dörring J and Torner L 1999 *Opt. Commun.* **159** 129
Mihalache D, Mazilu D, Crasovan L-C, Torner L, Malomed B A and Lederer F 2000 *Phys. Rev.* E **62** 7340
Torner L, Carrasco S, Torres J P, Crasovan L-C and Mihalache D 2001 *Opt. Commun.* **199** 277
[7] Soskin M S and Vasnetsov M V 2001 *Prog. Opt.* **42** 219
Soskin M S and Vasnetsov M V 1998 *Pure Appl. Opt.* **7** 301
[8] Kruglov V I and Vlasov R A 1985 *Phys. Lett.* A **111** 401
[9] Desyatnikov A S, Neshev D, Ostrovskaya E A, Kivshar Yu S, McCarty G, Krolikowski W and Luther-Davies B 2002 *J. Opt. Soc. Am.* B **19** 586
[10] Soljačić M, Sears S and Segev M 1998 *Phys. Rev. Lett.* **81** 4851
Soljačić M and Segev M 2000 *Phys. Rev.* E **62** 2810
Soljačić M and Segev M 2001 *Phys. Rev. Lett.* **86** 420
[11] Desyatnikov A S and Kivshar Yu S 2001 *Phys. Rev. Lett.* **87** 033901
[12] Desyatnikov A S and Kivshar Yu S 2002 *Phys. Rev. Lett.* **88** 053901
Desyatnikov A S and Kivshar Yu S 2002 *J. Opt. B: Quantum Semiclass. Opt.* **4** S58
[13] Rozas D, Law C T and Swartzlander G A 1997 *J. Opt. Soc. Am.* B **14** 3054
[14] Firth W J and Skryabin D V 1997 *Phys. Rev. Lett.* **79** 2450
Skryabin D V and Firth W J 1998 *Phys. Rev.* E **58** 3916
[15] Torner L and Petrov D V 1997 *Electron. Lett.* **33** 608
Torres J P, Soto-Crespo J M, Torner L and Petrov D V 1998 *J. Opt. Soc. Am.* B **15** 625
[16] Di Trapani P, Chinaglia W, Minardi S, Piskarskas A and Valiulis G 2000 *Phys. Rev. Lett.* **84** 3843
[17] Musslimani Z H, Soljačić M, Segev M and Christodoulides D N 2001 *Phys. Rev.* E **63** 066608
[18] Mihalache D, Mazilu D, Towers I, Malomed B A and Lederer F 2002 *J. Opt. A: Pure Appl. Opt.* **4** 615
[19] Mihalache D, Mazilu D, Crasovan L-C, Malomed B A and Lederer F 2000 *Phys. Rev.* E **62** R1505
[20] Mihalache D, Mazilu D, Crasovan L-C, Towers I, Buryak A V, Malomed B A, Torner L, Torres J P and Lederer F 2002 *Phys. Rev. Lett.* **88** 073902
Mihalache D, Mazilu D, Towers I, Malomed B A and Lederer F 2003 *Phys. Rev.* E **67** 056608
[21] Mihalache D, Mazilu D, Crasovan L-C, Towers I, Malomed B A, Buryak A V, Torner L and Lederer F 2002 *Phys. Rev.* E **66** 016613
[22] Desyatnikov A, Maimistov A and Malomed B 2000 *Phys. Rev.* E **61** 3107
[23] Mihalache D, Mazilu D, Crasovan L-C, Malomed B A and Lederer F 2000 *Phys. Rev.* E **61** 7142
[24] Vladimirov A G, McSloy J M, Skryabin D V and Firth W J 2002 *Phys. Rev.* E **65** 046606
Skryabin D V and Vladimirov A G 2002 *Phys. Rev. Lett.* **89** 044101
[25] Ruostekoski J and Anglin J R 2001 *Phys. Rev. Lett.* **86** 3934
Battye R A and Sutcliffe P M 2001 *Phys. Rev. Lett.* **86** 3989
[26] Battye R A, Cooper N R and Sutcliffe P M 2002 *Phys. Rev. Lett.* **88** 080401
[27] Savage C M and Ruostekoski J 2003 *Phys. Rev. Lett.* **91** 010403
[28] Williams J E and Holland M J 1999 *Nature* **401** 568
[29] Matthews M R, Anderson B P, Haljan P C, Hall D S, Wieman C E and Cornell E A 1999 *Phys. Rev. Lett.* **83** 2498
[30] Madison K W, Chevy F, Wohlleben W and Dalibard J 2000 *Phys. Rev. Lett.* **84** 806
[31] Abo-Shaeer J R, Raman C, Vogels J M and Ketterle W 2001 *Science* **292** 476
[32] Engels P, Coddington I, Haljan P C, Schweikhard V and Cornell E A 2003 *Phys. Rev. Lett.* **90** 170405
[33] Strecker K E, Partridge G W, Truscott A G and Hulet R G 2002 *Nature* **417** 150
Khaykovich L, Schreck F, Ferrari G, Bourdel T, Cubizolles J, Carr L D, Castin Y and Salomon C 2002 *Science* **296** 1290
[34] Saito H and Ueda M 2003 *Phys. Rev. Lett.* **90** 040403
[35] Abdullaev F Kh, Caputo J G, Kraenkel R A and Malomed B A 2003 *Phys. Rev.* A **67** 013605
[36] Baizakov B B, Malomed B A and Salerno M 2003 *Europhys. Lett.* **63** 642
[37] Crasovan L-C, Vekslerchik V, Pérez-García V M, Torres J P, Mihalache D and Torner L 2003 *Phys. Rev.* A **68** 063609
[38] Drummond P D 2000 *Mod. Phys. Lett.* B **14** 189
[39] Crasovan L-C, Molina-Terriza G, Torres J P, Torner L, Pérez-García V M and Mihalache D 2002 *Phys. Rev.* E **66** 036612
[40] Theocharis G, Frantzeskakis D J, Kevrekidis P G, Malomed B A and Kivshar Yu S 2003 *Phys. Rev. Lett.* **90** 120403
[41] Crasovan L-C, Kartashov Y V, Mihalache D, Torner L, Kivshar Yu S and Peréz-García V M 2003 *Phys. Rev.* E **67** 046610
[42] Peréz-García V M and Vekslerchik V 2003 *Phys. Rev.* E **67** 061804
[43] Desyatnikov A, Denz C and Kivshar Yu 2004 *J. Opt. A: Pure Appl. Opt.* **6** 209–12
[44] Julienne P S, Burnett K, Band Y B and Stwalley W C 1998 *Phys. Rev.* A **58** R797
[45] Heinzen D J, Wynar R, Drummond P D and Kheruntsyan K V 2000 *Phys. Rev. Lett.* **84** 5029
[46] Javanainen J and Mackie M 1998 *Phys. Rev.* A **59** R3186
[47] Duan L-M, Sorensen A, Cirac J I and Zoller P 2000 *Phys. Rev. Lett.* **85** 3991
[48] Cusack B J, Alexander T J, Ostrovskaya E A and Kivshar Yu S 2001 *Phys. Rev.* A **65** 013609
[49] Kokkelmans S J J M F and Holland M J 2002 *Phys. Rev. Lett.* **89** 180401
[50] Wynar R, Freeland R S, Han D J, Ryu C and Heinzen D J 2000 *Science* **287** 1016
[51] Donley E A, Claussen N R, Thompson S T and Wieman C E 2002 *Nature* **417** 529
[52] Torner L, Torres J P, Petrov D V and Soto-Crespo J M 1998 *Opt. Quantum Electron.* **30** 809
[53] Petrov D V, Torner L, Martorell J, Vilaseca R, Torres J P and Cojocaru C 1998 *Opt. Lett.* **23** 1444
[54] Minardi S, Molina-Terriza G, Di Trapani P, Torres J P and Torner L 2001 *Opt. Lett.* **26** 1004
[55] Kartashov Y V, Molina-Terriza G and Torner L 2002 *J. Opt. Soc. Am.* B **19** 2682
[56] Kartashov Y V, Crasovan L-C, Mihalache D and Torner L 2002 *Phys. Rev. Lett.* **89** 273902
[57] Towers I, Buryak A V, Sammut R A and Malomed B A 2001 *Phys. Rev.* E **63** 055601(R)
[58] Quiroga-Teixeiro M and Michinel H 1997 *J. Opt. Soc. Am.* B **14** 2004
Michinel H, Campo-Táboas J, Quiroga-Teixeiro M L, Salgueiro J R and García-Fernández R 2001 *J. Opt. B: Quantum Semiclass. Opt.* **3** 314

[59] Berezhiani V I, Skarka V and Aleksić N B 2001 *Phys. Rev.* E **64** 057601
[60] Towers I, Buryak A V, Sammut R A, Malomed B A, Crasovan L-C and Mihalache D 2001 *Phys. Lett.* A **288** 292
[61] Pego R L and Warchall H A 2002 *J. Nonlinear Sci.* **12** 347
[62] Malomed B A, Crasovan L-C and Mihalache D 2002 *Physica* D **161** 187
[63] Davydova T A, Yakimenko A I and Zaliznyak Y A 2003 *Phys. Rev.* E **67** 025402
[64] Mihalache D, Mazilu D, Crasovan L-C, Malomed B A, Lederer F and Torner L 2003 *Phys. Rev.* E **68** 046612
[65] Smektala F, Quemard C, Couderc V and Barthélémy A 2000 *J. Non-Cryst. Solids* **274** 232
Zhan C, Zhang D, Zhu D, Wang D, Li Y, Li D, Lu Z, Zhao L and Nie Y 2002 *J. Opt. Soc. Am.* B **19** 369
Boudebs G, Cherukulappurath S, Leblond H, Troles J, Smektala F and Sanchez F 2003 *Opt. Commun.* **219** 427
[66] Wright E M, Lawrence B L, Torruellas W and Stegeman G 1995 *Opt. Lett.* **20** 2481
Gatz S and Herrmann J 1992 *IEEE J. Quantum Electron.* **28** 1732
Enns R H, Rangnekar S S and Kaplan A E 1987 *Phys. Rev.* A **35** 466
Mihalache D, Mazilu D, Bertolotti M and Sibilia C 1988 *J. Opt. Soc. Am.* B **5** 565
[67] Malomed B A 1998 *Phys. Rev.* E **58** 7928